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PATHOLOGY IN DIVISOR CLASS GROUPS OF HYPERSURFACES

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In this article we study the variance in the number of generators of a divisorial ideal I versus two of its symbolic powers, namely I^{-1} and $I^{(2)}$. Perhaps it is not particularly surprising that the difference in the number of generators between I and I^{-1} can be made arbitrarily large for a hypersurface ring $A=R/(f)$, however it appears somewhat more remarkable that the number of generators of $I^{(2)}$ can be made arbitrarily large for I a two-generated prime divisorial in a normal domain A which is a normal quadratic extension of a regular local ring. We exhibit the latter phenomena in Section 1 by establishing a method of representing a rank two reflexive module over a regular local ring R as a divisorial ideal in a normal quadratic extension of R. The analytic spread of these ideals is also considered and, as a result, we obtain fairly detailed information concerning the ordinary powers versus symbolic powers. In Section 2 we establish the phenomena concerning divisorial ideals and their inverses. This is for the most part accomplished by appealing to ideas of M. Hochester expressed in [14]. As a consequence we demonstrate how this phenomena can provide an obstruction in obtaining a Noether normalization theorem for normal extensions.

We remark that an excellent reference for our terminology and notation with regard to divisor class groups is R. Fossum's book [10]. By a normal extension A of the normal domain R, we mean that the extension of fraction fields is Galois with group G, that A is the integral closure of R in the field extension (so $A^G=R$) and that the order of G is a unit in R (see [5]). If I is an ideal of the normal domain A, then

$$I^{**}=\operatorname{Hom}_A(\operatorname{Hom}_A(I,A),A)$$

is the associated divisorial ideal of A. Hence the divisor class group of A, denoted $\mathscr{C}\ell\, A$, consists of isomorphism classes of reflexive ideals with the operation defined

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by

$$[I]\cdot[J]=[(I\otimes J)^{**}],$$

where $[\cdot]$ denotes the isomorphism class of the respective divisorial ideals. Thus $[A]$ represents the multiplicative identity in $\mathscr{C}\ell A$, while $[I]^{-1}=[I^*]$. We shall occasionally abuse the notation by writing $I^{-1}=I^*$. If P is a prime divisorial ideal in A (the divisor class group of A is generated by such ideals), then the square of $[P]$ in $\mathscr{C}\ell A$ can be computed in two ways. From the preceding discussion we observe that

$$[P]^2=[(P\otimes P)^{**}]=[\mathrm{Hom}(P,P^*)^*].$$

However it is also the case that $[P]^2=[P^{(2)}]$ where $P^{(2)}=A\cap(P^2A_P)$. In general the nth symbolic power of P represents the nth power of P in the divisor class group. Finally, if G is the Galois group for the extension A of R and if I is a divisorial ideal of A, then I^σ denotes the image of I under the automorphism σ, $\sigma\in G$.

1. Two-generated divisorial ideals in normal quadratic extensions of regular local rings

Before venturing into the specifics of this section, we note that two-generated ideals in a normal domain play a fundamental role. For example, as noted by Buchsbaum and Eisenbud [4] (see also Evans and Griffith [8, Theorem 4.3]), if every two-generated ideal in a local domain A has finite projective dimension, then A is factorial (i.e. a unique factorization domain). Furthermore, every divisorial ideal I in a normal domain is of the form $I=(x,y)^{**}$ for some x,y in I. We will consider the special situation in which (x,y) is a prime divisorial ideal.

In order to assure that normal extensions are local we will assume throughout this section (unless otherwise mentioned) that R is a complete regular local ring. We also assume that R contains an infinite field of characteristic other than two, i.e., $1/2\in R$. Let E be any reflexive R-module of rank two. From Bruns [2] (or [8, Corollary 3.13]) such an E fits into an exact sequence $0\to E\to R^3\to(a,b,c)\to 0$, where the ideal (a,b,c) has height at least two. If (a,b,c) has height three, we remark that E cannot be free since (a,b,c) will have projective dimension two. In fact E must be the second syzygy in a Koszul complex for $R/(a,b,d)$. If the height of (a,b,c) is two, then we know from [7, Theorem 4.4] that E is free if and only if (a,b,c) is unmixed. Since R contains an infinite field we may assume by Flenner's version of Bertini's Theorem [9] that each of a,b and c are principal primes in R. We further want to make one additional normalization assumption. It will follow as a consequence of our first lemma.

Lemma 1.1. *Let R be as above and let (a,b,c) be an ideal of height two in R in which a, b and c are principal primes. By enlarging the field in R we may also assume that $a^2+bc=p$ is a principal prime in R.*

Proof. By grading $R[X, Y, Z]$ so that X, Y and Z have degree one while the elements of R have degree zero, it is straightforward to check that the homogeneous element $(aX)^2+(bY)(cZ)=a^2X^2+bcYZ=p$ of degree two in $R[X, Y, Z]$ is a principal prime. After increasing a coefficient field of R to contain all polynomials $f(X, Y, Z)$ with coefficients in the coefficient field, the ideal with generators aX, bY and cZ has the desired property.

At this point we simplify our notation and assume that a, b and c have the properties of aX, bY and cZ of Lemma 1.1. We also note that this base change has no effect on the homological properties of E. We will keep the preceding notation throughout this section.

In the endomorphism ring of R^2 over R, there is an endomorphism ε whose matrix is given by

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

Somewhat abusively, we also refer to this matrix as ε. We note that ε has trace equal to zero and that $\varepsilon^2=(a^2+bc)\cdot 1=p\cdot 1$. Consequently, identifying R with the center of $\operatorname{End} R^2$, we see that $R[\varepsilon]$ is a commutative subring which is isomorphic to $R[T]/(T^2-p)$, with p a prime in R. Therefore $A=R[\varepsilon]$ is a normal domain (see [10, Lemma 11.1]). It is also a normal extension of R of degree two (see the introduction for a defintion) with $\mathbb{Z}_2$-action defined by $x+y\sqrt{p}\to x-y\sqrt{p}$. Moreover A is local and complete since R is.

Let $E'=\{\phi\in\operatorname{End} R^2 \mid \operatorname{tr}\phi=\operatorname{tr}(\varepsilon\phi)=0\}$. With a basis of R^2 having been chosen such that

$$\varepsilon=\begin{bmatrix} a & b \\ c & -a \end{bmatrix},$$

we see that

$$\phi=\begin{bmatrix} x & y \\ z & -x \end{bmatrix}$$

is in E' if and only if $2ax+bz+cy=0$, that is, if and only if $\langle 2x, z, y\rangle\in R^2$ maps to zero in the above presentation $R^3\to(a,b,c)$. It follows that

$$\phi=\begin{bmatrix} x & y \\ z & -x \end{bmatrix} \text{ is in } E' \quad\Leftrightarrow\quad \langle 2x, z, y\rangle \text{ in } E.$$

Thus, since $1/2\in R$, we have that E' is isomorphic to E as an R-module. This part of the construction is similar to one considered by the author in [12]. Further, if $\alpha=r_1\varepsilon+r_0 1$ represents an element of $R[\varepsilon]$ and if ϕ in E', then it is straightforward that $\alpha\phi\in E'$. Thus E' (and hence also E) becomes an A-module under this action. Since A is finitely generated as an R-module, it follows that E is a reflexive A-module since it is reflexive as an R-module. Finally, the fact that both A and E have rank two as an R-module shows that the rank of E as an A-module must be one. Thus E is isomorphic to a divisorial A-ideal and therefore represents an element in $\mathscr{C}\ell\, A$; Before looking further into the structure of E as a divisorial A-module, we establish a general fact about the relations on two generated divisorial ideals in Gorenstein normal domains.

Lemma 1.2. *Let A be a Gorenstein normal domain of dimension at least two and let I be a Cohen–Macaulay ideal (necessarily divisorial) which is generated by two elements. Then I^*, the inverse of I in $\mathscr{C}\ell\, A$, is the first syzygy of I. If in addition A/I is Gorenstein, then I^* is also two-generated, and the free resolution of I is periodic of period two with every odd syzygy being I^* and every even one being I.*

Proof. Let $I=(x,y)$ and consider the exact sequence $0 \to A \to I \to C \to 0$ where 1 in A is sent to x and $C=A/K$. Hence $K=\{a \in A \mid ay \in (x)\}$. Now $\dim C = \dim A - 1$ and $\operatorname{depth} C = \dim A - 1$ since both A and I are Cohen–Macaulay. Therefore C is Cohen–Macaulay of dimension one less than A. Dualizing the above short exact sequence with respect to A gives the exact sequence $0 \to I^* \to A^* \to \operatorname{Ext}^1(C,A) \to 0$ since $\operatorname{Ext}^1(I,A)=0$ (this follows from the fact that I is Cohen–Macaulay and A is Gorenstein). Moreover, this shows that the dualizing module $\Omega_C^0 = \operatorname{Ext}^1(C,A)$ is isomorphic to C since it is cyclic and both C and Ω_C^0 are Cohen–Macaulay. Thus, identifying A^* with A, we have that $A/I^* \cong A/K$. We obtain from Schanuel's Lemma that $A \oplus I^* \cong A \oplus K$ and hence, since A is local, that $I^* \cong K$. From the pullback of the two maps $I \to A/I^* = C$ and $A \to A/I^* = C$ we obtain a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & I^* \arrow[r, equal] \arrow[d] & I^* \arrow[d] & \\
0 \arrow[r] \arrow[d, dash] & A \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & I \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. It follows from the middle row that $P \cong A^2$ and that I^* is the first syzygy of I. In case A/I is Gorenstein, one need only to dualize the short exact sequence $0 \to I \to A \to A/I \to 0$ to see that I^* is also two-generated.

Our next result describes exactly how E is situated in the divisor class group of A. The original proof of this result from a somewhat different point of view was communicated to me by Craig Huneke.

Theorem 1.3 (Notation as above). *Let I be the A-ideal, $I = (\sqrt{p} + a, b)$, and let σ be the generator of the $\mathbb{Z}_2$-action on A.*

Then I is a Cohen–Macaulay prime ideal of A of height one. The ideal $I^\sigma = (-\sqrt{p} + a, b)$ represents the inverse of I in $\mathscr{C}\ell\, A$. Moreover, there is an A-isomorphism

$$E \cong \mathrm{Hom}_A(I^\sigma, I)$$

which shows that E is A-isomorphic to $I^{(2)}$. That is, E represents the square of I in the divisor class group of A.

Proof. That I is a Cohen–Macaulay prime ideal in A of height one follows from the fact that R and A have the same dimension together with the isomorphism $A/I \cong R/(b)$. We recall that b is a prime element in the regular local ring R. It is routine to establish that $I \cap R = I^\sigma \cap R = (b)$ and that $I \cap I^\sigma = bA$, since b is a prime in R and since the only primes in A that contract to (b) are I and I^σ (see [17, p. 33]). It follows that I^σ is the inverse of I in $\mathscr{C}\ell\, A$ and hence that I^σ is A-isomorphic to I^*. From one point of view of the operation on $\mathscr{C}\ell\, A$ (see Introduction) we see that

$$(I^{(2)})^* = (I^*)^{(2)} \cong \mathrm{Hom}_A(I^* \otimes_A I^*, A)^* \cong \mathrm{Hom}_A(I^*, I)^*.$$

Therefore, since $I^* \cong I^\sigma$, it follows that the inverse of $I^{(2)}$ is the inverse of $\mathrm{Hom}_A(I^\sigma, I)$. Hence our claim $E \cong I^{(2)}$ will follow provided we can establish the A-isomorphism $E \cong \mathrm{Hom}_A(I^\sigma, I)$.

We remark that $\{-\sqrt{p} + a, b\}$ and $\{\sqrt{p} + a, b\}$ form R-bases for the free R-modules I^σ and I, respectively. Thus an A-homomorphism $f : I^\sigma \to I$ is determined by a two by two matrix over R which admits scalar multiplication by elements of A. So let

$$f(-\sqrt{p} + a) = u((\sqrt{p} + a) + vb,$$

$$f(b) = r(\sqrt{p} + a) + sb$$

where u, v, r, s in R determine the matrix

$$\begin{bmatrix} u & r \\ v & s \end{bmatrix}$$

mentioned above. The relations on the generators of I are by Lemma 1.2:

(i) $(\sqrt{p}+a)(-\sqrt{p}+a)+cb=0$,

(ii) $-bx+bx=0$,

where x equals $-\sqrt{p}+a$ or x equals b. In other words, there is an exact sequence $0\to I\to A^2\to I^\sigma\to 0$. From these relations which f must respect we obtain the equations

$$(\sqrt{p}+a)\,f(-\sqrt{p}+a)+c\,f(b)=0$$

or

$$(\sqrt{p}+a)[u(\sqrt{p}+a)+vb]+c[r(\sqrt{p}+a)+sb]=0$$

or

$$u(\sqrt{p}+a)^2+v(\sqrt{p}+a)+cr(\sqrt{p}+a)+csb=0.$$

Since the relation $p=a^2+bc$ gives that $p+2a\sqrt{p}+a^2=2a(\sqrt{p}+a)+cb$, the last equation above becomes

$$(2au+bv+cr)(\sqrt{p}+a)+(uc+cs)b=0$$

in I. Since $\{\sqrt{p}+a,b\}$ is an R-basis for I, it follows that $2au+bv+cr=0$ and that $-u=s$, that is, the matrix determined by f must be of the form

$$\begin{bmatrix} u & r \\ v & -u \end{bmatrix},$$

where $2au+bv+cr=0$. Furthermore, reversing the preceding arguments demonstrates that any such matrix yields an A-homomorphism $I^\sigma\to I$. Thus each element f represents an element in the module E' (which is R-isomorphic to E) as described above. To show that the R-isomorphism ψ (as described above) from $\mathrm{Hom}_A(I^\sigma,I)$ to E' is an A-isomorphism, it suffices to show that $\psi(\sqrt{p}f)=\sqrt{p}\psi(f)$. However, in viw of the above description of f, this is a straightforward calculation which uses the facts that $2au+bv+cr=0$ and

$$\sqrt{p}\begin{bmatrix} u & r \\ v & -u \end{bmatrix}=\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} u & r \\ v & -u \end{bmatrix}.$$

Corollary 1.4 (Notation as above). *The symbolic square of the ideal* $(\sqrt{p}+a,b)$ *can need an arbitrarily large number of generators.*

Proof. Let $\dim R=n\geq 2$ and let K be the second syzygy in a Kozul complex for the residue field of R. Then $\operatorname{rank} K=n-1$ and K is minimally generated by $\binom{n}{2}=1/2n(n-1)$ elements. Using Bruns' result [2] (see also [8, Theorem 3.11]) there is a free submodule F or rank $n-3$ of K so that $K/F=E$ is a rank two reflexive R-module which is minimally generated by $\frac{1}{2}n(n-1)-(n-3)=\frac{1}{2}(n^2-3n+6)$ elements. Since A is free of rank two over R we observe that $\mu_R(E)\leq 2\mu_A(E)$, where $\mu_R(E)$ and $\mu_A(E)$ denotes the size of a minimal generating set for E over R and A, respectively. Consequently, as an A-module $\mu_A(E)\geq\frac{1}{4}(n^2-3n+6)$, which gets arbitrarily large as $n=\dim R$ does.

Now we should like to look further into the structure of the other symbolic and ordinary powers of I in A. Although we cannot explicitly describe this family of rank two reflexive modules induced by E in this representation, we do achieve a description of some properties.

From Huneke's article [15, 1.10] we have that $I=(\sqrt{p}+a, b)$ is generated by a D-sequence. This suggests that the Rees algebra of I over A should reveal information on the structure of the powers of I.

Theorem 1.5 (Notation as above). *The ideal $I=(\sqrt{p}+a, b)$ has the following properties.*

(i) *I is unramifed.*
(ii) *The analytic spread of I is two.*
(iii) *The first syzygy of I^n is a direct sum of copies of I^* for each $n \geq 1$.*
(iv) *$I^n \neq I^{(n)}$ for each $n \geq 2$.*
(v) depth $I^n = \dim A - 1$ *for each $n \geq 2$.*

Proof. Part (i) follows from the fact that $\sqrt{p} \notin I$. The Rees Algebra of I over A is $\mathscr{R} = A \oplus IT \oplus I^2T^2 \oplus \cdots$ and the analytic spread of I is the dimension of $\mathscr{R}/\mathfrak{m}_A \mathscr{R}$, where $\mathfrak{m}_A$ is the maximal ideal of A. Since I is two generated it follows that its analytic spread is one or two. By [16, Lemma 2.2] the analytic spread cannot be one since I is not principal. Thus the analytic spread of I is two and $\mathscr{R}/\mathfrak{m}_A \mathscr{R} \cong \mathbf{k}[X, Y]$, where $\mathbf{k}$ is the residue field of A. Hence part (ii) is established.

In order to verify part (iii) we note there is a surjective ring homomorphism of $A[X, Y]$ onto the Rees algebra $\mathscr{R}$, by sending X to $(\sqrt{p}+a)T$ and Y to bT. Let Q be the kernel of this homomorphism. Computing the change in dimension we get that Q is a prime ideal of height one in $A[X, Y]$. Since there is a natural isomorphism $\mathscr{C}\ell\, A \to \mathscr{C}\ell\, A[X, Y]$ via the map $J \mapsto A[X, Y] \otimes_A J$, we have that $Q \cong J[X, Y]$ for some divisorial ideal J of A. Therefore we have an exact sequence

$$0 \to J[X, Y] \to A[X, Y] \to \mathscr{R} \to 0,$$

which viewed as an A-sequence, gives the first syzygy of the A-module $I \oplus I^2 \oplus I^3 \oplus \cdots$. By Lemma 1.2 the first syzygy of I is I^*. Hence I^* must be a summand of $J[X, Y]$ as an A-module. However, $J[X, Y]$ is isomorphic to a direct sum of an infinite number of copies of J as an A-module. Since $\mathrm{End}_A J \cong A$ is local, we may apply the Krull–Schmidt Theorem to obtain that J is isomorphic to I^*.

This gives part (iii) and, in addition, that depth $I^n \geq \dim A - 1$ for $n \geq 1$. Now suppose that I^n is Cohen–Macaulay for some $n > 1$. Then $I^{(n)} = I^n$ and the minimal free A-presentation,

$$0 \to \bigoplus^{h} I^* \to \bigoplus^{h+1} A \to I^n \to 0,$$

of I is dual exact. That is, the A-dual sequence

$$0 \to (I^n)^* \to \bigoplus^{h+1} A^* \to \bigoplus^{h} I \to 0$$

is exact. But then it must be that $I^n = I^{(n)} \cong I$ and, from the minimality, that $h=1$. It follows that I^n is two-generated. However, the fact that $\mathscr{R}/\mathrm{m}_A\,\mathscr{R} \cong \mathbf{k}[X, Y]$ gives that $\mu_A(I^n) = n+1$. In particular, we have $\mu_A(I^n) > 2$ for $n \geq 2$. Thus I^n cannot be Cohen–Macaulay for $n > 1$ and, therefore, depth $I^n = \dim A - 1$ for $n > 1$.

To see that $I^n \neq I^{(n)}$ for each $n > 1$ we apply Huneke's result [15, Theorem 3.1]. We can find a prime ideal P in A of height two such that $I_P^{(2)}$ is not principal and, as above, has analytic spread equal to two over A_P. Actually one can take P to be any prime in A of height two which contains the ideal (a, b, c). As previously noted we have $\mu(I_P^n) = n+1$ for $n > 1$. However, the reflexive ideals $I_P^{(n)}$ are rank two reflexive modules over the two-dimensional regular local ring $R_{P \cap R}$ and therefore are free over $R_{P \cap R}$. It follows that $\mu(I^{(n)}) \leq 2$ for $n > 1$, and that $I_P^n \neq I_P^{(n)}$ for $n > 1$. This concludes the argument for part (iv) and completes the proof of 1.5.

Two generic examples emerge from the above construction, namely the generic R-sequence of length three and the generic three-generated ideal of height two.

Example 1.6. *The generic R-sequence of length three.* Let R be the power series ring $R = \mathbf{k}[[X, Y, Z]]$ with $\mathbf{k}$ a field of characteristic not two. Let E be the first syzygy of the three-generated ideal (X, Y, X), i.e., E is the second syzygy of the residue field $\mathbf{k}$. We note that the endomorphism

$$\varepsilon = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$$

satisfies $\varepsilon^2 = (X^2 + YZ) \cdot 1$ and that $p = X^2 + YZ$ is a prime in R. Hence E is a divisorial ideal for the local ring

$$A = R[\varepsilon] \cong \mathbf{k}[[X, Y, Z]][T]/(T^2 - X^2 - YZ)$$

which is also isomorphic to the factor ring $\mathbf{k}[[X, Y, Z, T]]/(T^2 - X^2 - YZ)$. Furthermore, with $U = T - X$, $V = T + X$ we see that A is isomorphic to the well known (completed) hypersurface $\mathbf{k}[[U, V, Y, Z]]/(UV - YZ)$. In this case one notes that E is three-generated and represents the square of the divisorial ideal (v, y) in $A = \mathbf{k}[[U, V, Y, Z]]/(UV - YZ)$, where the lower case letters represent the respective variables modulo the relation $UV - YZ = 0$. The divisor class group of A is $\mathbb{Z}$ and so the order of E is infinite.

Example 1.7. The *generic three-generated prime ideal of height two.* Let $\mathbf{k}$ be a field of characteristic not two and let $\alpha, \beta, \gamma, x, y, z$ be indeterminates over $\mathbf{k}$. We let $R = \mathbf{k}[\alpha, \beta, \gamma, x, y, z]$ and consider a graded version of our construction which is analogous to the local case. Our three generated ideal will be the one generated by the 2×2 minors of the 2×3 matrix

$$\begin{bmatrix} \alpha & \beta & \gamma \\ x & y & x \end{bmatrix}.$$

Let $\Delta_1 = \beta z - \gamma y$, $\Delta_2 = -(\alpha z - \gamma x)$ and $\Delta_3 = (\alpha y - \beta x)$. With Δ_1, Δ_2 and Δ_3 playing the roles of b, a and c, respectively (from our earlier notation), then it is straightforward to check that $\Delta_2^2 + \Delta_1\Delta_3$ is a principal prime in R. We let $A = R[T]/(T^2 - \Delta_2^2 - \Delta_1\Delta_3)$ in keeping with our construction. By giving the variable T a degree of two while the variables $\alpha, \beta, \gamma, x, y, z$ have the usual degree of one, we may consider A a graded normal domain. The divisor class group of A will coincide with that of its localization at the maximal ideal $(\alpha, \beta, \gamma, x, y, z, T)$. We wish to compute this divisor class group. To accomplish this end let $B_0 = \mathbf{k}[\Delta_1, \Delta_2, \Delta_3, T]/(T^2 - \Delta_2^2 - \Delta_1\Delta_3)$. This normal domain occurred in the previous example and is well known to have divisor class group $\mathbb{Z}$. Then $B = B_0[z, \gamma, x]$ also has divisor class group $\mathbb{Z}$ since z, γ, x are algebraically independent over B. Moreover, the rings B and A are birational since $\alpha = 1/z(\gamma x - \Delta_2)$, $y = -1/\Delta_2(x\Delta_1 + z\Delta_3)$ and $\beta = -1/\Delta_2(\alpha\Delta_1 + \gamma\Delta_3)$. In particular we have

$$A[z^{-1}, \Delta_2^{-1}] = B[z^{-1}, \Delta_2^{-1}].$$

Since the relation $p = a^2 + bc$ gives that $p + 2a\sqrt{p} + a^2 = 2a(\sqrt{p} + a) + cb$, the last equation above becomes

In the above example the divisorial ideal I and its symbolic square are both two-generated and Cohen–Macaulay. That $I^{(2)}$ is two-generated follows from the fact that the syzygy E over R for the ideal $(\Delta_1, \Delta_2, \Delta_3)$ is free of rank two. Two final questions arise. Does A always have divisor class group $\mathbb{Z}$? (or at the very least does E always have infinite order?) Secondly, what is the free locus of $I^{(n)}$ over R, for $n > 2$, in terms of the free locus of $I^{(2)}$ over R? (e.g. if $I^{(2)}$ represents a vector bundle on the puncture spectrum of R with $\dim R > 4$, can the $I^{(n)}$ also be bundles for $n \gg 0$?).

2. The number of generators of the inverse of a divisorial ideal

As in the previous section we will be dealing for the most part with normal domains which have divisor class group isomorphic to $\mathbb{Z}$. In order to emphasize the difference between normal and nonnormal extensions, we begin with a discussion of normal extensions.

Theorem 2.1. *Let A be a normal extension of a factorial domain R and let G be the group of R-automorphisms of A. Then the fixed subgroup $(\mathscr{C}\ell\, A)^G$ of $\mathscr{C}\ell\, A$ is a torsion group.*

Proof. The action of G on $\mathscr{C}\ell\, A$ is defined by $[I] \mapsto [I^\sigma]$ for $\sigma \in G$. Now suppose that $[I]$ is fixed under this action of G. Then I is A-isomorphic to I^σ for each $\sigma \in G$. Let g be the order of G. Then $I^{(g)} \cong \bigcap_{\sigma \in G} I^\sigma$ follows from the preceding statement together with the definition of product (via intersections) in $\mathscr{C}\ell\, A$. It follows that $I^{(g)}$ is G-invariant and that $[I]$ is a torsion element if and only if $[I^{(g)}]$ is. Thus we have reduced the question to the case of a G-invariant ideal.

We now assume the ideal I is G-invariant. Then I can be written as the intersection $I = I_1 \cap I_2 \cap \cdots \cap I_m$, where each I_j consists of intersections of symbolic powers of height one primes in the same conjugacy class under the action of G, the conjugacy classes being distinct for different indices. Since I is G-invariant so are the I_j. Hence in order to show that I represents an element of finite order in $\mathscr{C}\ell\, A$, it suffices to do this in case I is G-invariant and $I = P_1^{(l_1)} \cap \cdots \cap P_s^{(l_s)}$, where each P_i and P_j are conjugate primes under the action of G. Since I is G-invariant, it follows that $l_1 = l_2 = \cdots = l_s$ and that each conjugate of P_1 appears in the above primary decomposition. It then follows that I represents the lth power of the ideal $P_1 \cap \cdots \cap P_s$ in $\mathscr{C}\ell\, A$. Now let $(b) = P_1 \cap R$. By [17, pp. 33, 34] any two primes in A which lie over the prime (b) are conjugate under the action of G. Hence, the G-invariant principal A-ideal bA has a primary decomposition $bA = P_1^{(e)} \cap \cdots \cap P_s^{(e_s)}$. This equation simply shows that the ideal $P_1 \cap \cdots \cap P_s$ represents an element of order e in $\mathscr{C}\ell\, A$. Consequently I also represents an element of finite order in $\mathscr{C}\ell\, A$.

Corollary 2.2 (Same hypothesis as 2.1). *If $\mathscr{C}\ell\, A \cong \mathbb{Z}$, then each divisorial ideal is G-conjugate to its inverse in $\mathscr{C}\ell\, A$.*

Proof. As previously observed, there is an action of G on $\mathscr{C}\ell\, A$. Since $\mathscr{C}\ell\, A = \mathbb{Z}$ it follows from Theorem 2.1 that $(\mathscr{C}\ell\, A)^G = 0$. Thus the action of G on $\mathscr{C}\ell\, A = \mathbb{Z}$ is nontrivial. However the only nontrivial group action on $\mathbb{Z}$ is the one $n \to -n$. Thus, given any nontrivial divisorial ideal I in A, there is some $\sigma \in G$ such that $I^\sigma \cong I^*$.

Corollary 2.3. *Let A be a normal extension of the factorial domain R with group G such that $A^G = R$. If $\mathscr{C}\ell\, A = \mathbb{Z}$, then every divisorial ideal I has the same number of generators over A as its inverse I^*.*

Proof. Let I be a divisorial ideal of A and suppose I is generated by $x_1, \ldots, x_m$. Then $I^* \cong I^\sigma$ for some $\sigma \in G$ by Corollary 2.2. The conclusion follows from the fact that I^σ is minimally generated by $\sigma(x_1), \ldots, \sigma(x_m)$ over A.

In part the above discussion demonstrates the following point. That is, if A is a normal finitely generated $\mathbf{k}$-algebra, where $\mathbf{k}$ is a field, and if A contains a divisorial ideal I with the property that $\mu(I) \neq \mu(I^*)$, then A cannot be realized as a finite normal extension of a polynomial ring over $\mathbf{k}$. In particular, the property $\mu(I) \neq \mu(I^*)$ is an obstruction to strengthening the Noether normalization theorem to include the normal condition that A be the integral closure in a Galois extension of the fraction field of the polynomial ring. We will now supply a host of examples in which the difference of $\mu(I) - \mu(I^*)$ can be made arbitrarily large. We first will set up some notation.

Here $(R, \mathfrak{m}, \mathbf{k})$ will denote a regular local ring of dimension at least two and of equal characteristic. We denote by K the fraction field of R. Let P be a prime ideal in R of height two. Following closely the construction of Hochster [14] concerning grade reduction (especially for prime ideals of height larger than two), we shall construct a class of normal domains having $\mathbb{Z}$ as divisor class group and generally having properties contrary to normal extensions of regular (e.g. polynomial) rings.

Let the elements $a_1, \dots, a_t$ generate P. We may assume that a_1 and a_2 generate P locally at R_P. Let S be the full ring of polynomials $R[Y_1, \dots, Y_t]$ an let $\delta = \sum_{i=1}^{t} a_i Y_i$. Let A denote the factor ring of $R(Y_1, \dots, Y_t)$ modulo the ideal generated by δ.

Proposition 2.4. *The ring A is a normal domain having divisor class group $\mathbb{Z}$ generated by PA.*

The proof of the above proposition is accomplished via the properties summarized in the following lemma .

Lemma 2.5 (The notation is as above). (a) *If p is a principal prime in $R-P$, then p represents a principal prime in A.*

(b) *The ideal PA is a prime ideal of height one in A and, moreover, A localized at PA is a discrete valuation ring.*

(c) *The fraction field K of R has the property that $K \otimes_R A$ is a regular factorial domain.*

(d) *The ring A is a normal domain with divisor class group $\mathbb{Z}$ generated by PA.*

Proof (a). Since the grade of the R-ideal (P, p) in R is at least three, the fact that $(R/pR)[Y_1, \dots, Y_t]/(\sum_i \bar{a}_i Y_i)$ is a domain (where $\bar{a}_i = a_i + (p)$ in R/pR) follows from the main theorem (part b) of Hochster [14]. Therefore the same statement holds when round brackets replace square brackets.

(b). To see that PA is a prime ideal in A, one can observe that $A/PA \cong (R/P)(Y_1, \dots, Y_t)$. After localizing $S = R[Y_1, \dots, Y_t]$ at the prime ideal $P[Y_1, \dots, Y_t]$ one has that $Y_1, \dots, Y_t$ are invertible and that δ is a 'generic' generator of the localization of $P[Y_1, \dots, Y_t]$. Consequently, PA is a principal ideal after localization at PA. The fact that A is a domain follows once again from the main result of Hoschster [14].

(c). We note that $K \otimes_R A$ is a localization of the ring $K[Y_1, \dots, Y_t]/(\delta)$, where $\delta = a_1 Y_1 + \dots + a_t Y_t$ with $a_i \in K$. Hence $K \otimes_R A$ is a localization of a polynomial ring over K in $t-1$ variables.

(d). The fact that A is a normal domain follows from (a), (b) and (c). From (a) we can see that the divisor class group of A is isomorphic to that of $R_P[Y_1, \dots, Y_t]/(\delta)$. Since $PR_P = (a_1, a_2)$ we may obtain a new basis for the variables $Y_1, \dots, Y_t$, say $Y_1', \dots, Y_{t-1}'$ so that $\delta = a_1 Y_1' - a_2 Y_2'$. Then $R_P[Y_1, \dots, Y_t]/(\delta)$ is isomorphic to a polynomial ring over $R_P[Y_1', Y_2']/(a_1 Y_1' - a_2 Y_2')$. The completion of this latter ring at the ideal (a_1, a_2, Y_1', Y_2') is isomorphic to a power series ring of the form $L[[a_1, a_2, Y_1', Y_2']]/(a_1Y_1' - a_2Y_2')$, where L is a field. But this ring is a normal domain having divisor class group $\mathbb{Z}$ generated by (a_1, a_2) (see Section 1 for a discussion of this ring.) It follows that the ring $R_P[Y_1, \ldots, Y_t]/(\delta)$ has divisor class group $\mathbb{Z}$ generated by the image of P. This completes our argument.

In order to discuss further the behavior of the divisor class group of A we wish to consider the prime ideal P obtained in the following manner. Let R be the local ring at the origin of the polynomial ring $\mathbf{k}[X_1, \ldots, X_n]$, where $\mathbf{k}$ is an infinite field and $n \geq 4$. Let M be a finitely generated reflexive R-module such that $\mathrm{Ext}_R^1(M, R) = 0$ (e.g. if M is a second syzygy in a Koszul complex for $(X_1, \ldots, X_n)$). By the results [6], [18] we may obtain an exact sequence (see also Bourbaki [1, Chapter 7]) $0 \to R^s \to M \to P \to 0$ in which P is a prime ideal and an R-basis of R^s consists of s-generators of M. Since $\mathrm{Ext}_R^1(M, R) = 0$, we obtain an exact sequence

$$0 \to P^* \to M^* \to (R^s)^* \to \mathrm{Ext}_R^1(P, R) \to 0$$

after dualizing with respect to the ring R. Therefore, the numberof generators of $\mathrm{Ext}_R^1(P, R)$ over R is at most s. We also note that $\mathrm{Ext}_R^1(P, R) \cong \mathrm{Ext}_R^2(R/P, R)$ and that the dualizing module $\Omega_{R/P}^0 = \mathrm{Ext}_R^2(R/P, R)$ since P has height two (see [11], [13] for a discussion concerning the vanishing and nonvanishing Ext's in this situation).

Proposition 2.6 (The notation is as in 2.4 and 2.5). *The divisorial ideal which represents the inverse of PS in $\mathscr{C}\ell\, A$ can be generated by $1 + \mu$ elements where μ is the number of generators of the R-module* $\mathrm{Ext}_R^1(P, R) \cong \Omega_{R/P}^0$.

Proof. We use the fact that the ring A is Gorenstein. Since PA has height one in A, the dualizing module for A/PA is $\mathrm{Ext}_A^1(A/PA, A)$. However, $A/PA \cong (R/P)(Y_1, \ldots, Y_t)$ and so $\mathrm{Ext}_A^1(A/PA, A)$ is isomorphic to $\Omega_{R/P}^0(Y_1, \ldots, Y_t)$. Hence the number of generators of $\mathrm{Ext}_A^1(A/PA, A)$ over A is μ. Now dualizing the exact sequence $0 \to PA \to A \to A/PA \to 0$ with respect to A gives the exact sequence

$$0 \to A^* \to (PA)^* \to \mathrm{Ext}_A^1(A/PA, A) \to 0.$$

The number of generators of the divisorial A-ideal $(PA)^*$ is less than or equal to $1 + \mu$. The ideal $(PA)^*$ is one way of obtaining the inverse of PA in $\mathscr{C}\ell\, A$. The only remaining fact to clear up (so that we may apply the discussion just preceding this proposition) is that the prime P fits into an exact sequence $0 \to R^s \to M \to P \to 0$, in which M is a reflexive module satisfying $\mathrm{Ext}_R^1(M, R) = 0$. However, this fact follows from [3; Lemma 2.1(b)].

We now will discuss a class of specific examples. As above the ring R represents the local ring at the origin of a polynomial ring $\mathbf{k}[X_1, \ldots, K_n]$, where $\mathbf{k}$ is an infinite field and $n \geq 4$. Let M be the second syzygy in a Koszul complex for the ideal $(X_1, \ldots, X_n)$. Then M has $\binom{n}{2}$ generators. As described in [6] we construct an exact sequence

$$0 \to R^{n-2} \to M \to P \to 0$$

in which P is a prime ideal having $\binom{n}{2} - n + 2$ generators.

Corollary 2.7. *For the situation described above the difference in the number of generators of PA and its inverse in* $\mathscr{C}\ell\, A$ *is at least* $\binom{n}{2}-2n+3$. *Hence, for* $n\geq 4$ *the ring A cannot be a normal extension of a factorial Gorenstein domain.*

Proof. From Proposition 2.6 the ideal inverse of PA can be generated by $1+(n-2)=n-1$ generators. Thus the differences in the number of generators of PA less that of its ideal class inverse is at least

$$\binom{n}{2}-n+2-(n-1)=\binom{n}{2}-2n+3.$$

For $n\geq 4$, this difference is positive. We recall from Proposition 2.4 that the normal domain A has divisor class group $\mathbb{Z}$. By Corollary 2.3, if A were a normal extension of a factorial Gorenstein domain, then a divisorial ideal and its ideal class inverse necessarily are generated by the same number of elements.

Since the number $\binom{n}{2}=\frac{1}{2}(n^2-5n+6)=\frac{1}{2}(n-2)(n-3)$ gets arbitrarily large as n does, we observe the above examples illustrate that the difference in the number of generators between a divisorial ideal and its ideal class inverse can be made arbitrarily large.

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